Name:________________________________ October 11, 2013

Algebra II/Trig: Adjacency Matrices of Networks

1 Networks

First, imagine I have a network of some kind. A **network** just means any collection of points (**nodes**) that are connected in some way.

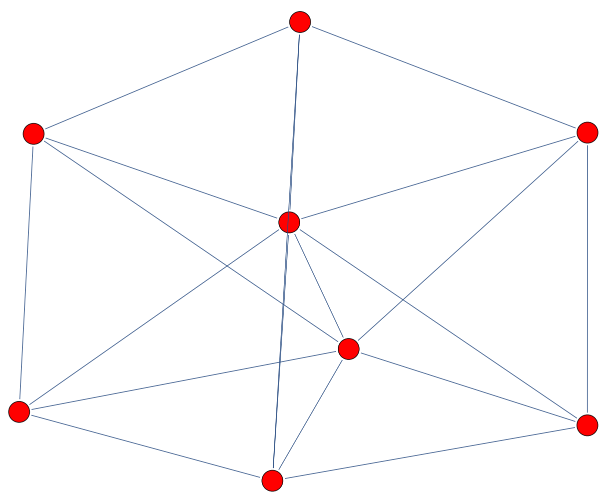


Figure 1: A made-up network.

That's a pretty vague definition, which is good, because that lets us apply this idea to a lot of systems, such as

- The **internet** is a network, with the webpages in red, and links to other pages as blue connections
- **IMDB** (Internet Movie Database) has which actors have been in movies together; here, the actors are the red nodes and the movies they're in together are the blue connections.
- The **flights** between different cities are also a network: the airports are the nodes and the planes are the connections

It's a pretty common problem is science in general to know how many jumps it takes you to get from one node to a different node.

2 Adjacency Matrix

Lets say I give you the following network with just four nodes:

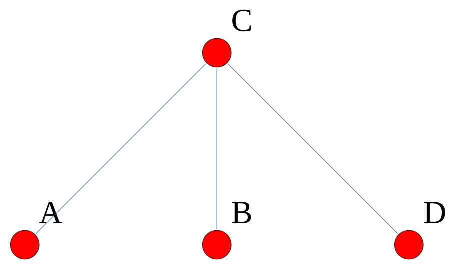


Figure 2: A made-up network with letters.

To write down the **Adjacency Matrix** of the network, you do the following:

1. Draw a square matrix with the same number of columns/rows as there are nodes in the graph
2. Write the names of the nodes at the top of each column and to the left of each row

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

3. If two nodes are connected, put a 1 at that point in the matrix. If they are not connected, put a zero. For instance, A and C are connected, so I would put a 1 there, but A is not connected to B or D, so I would put 0's there. So the A column and row would look like

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & & & \\ 1 & & & \\ 0 & & & \end{pmatrix} \end{array}$$

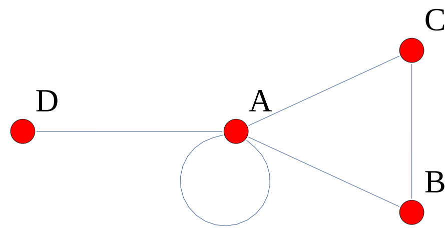
4. Fill in the whole matrix in this fashion. For the above graph, you would get:

$$\underline{\mathbf{A}} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

3 Some Practice

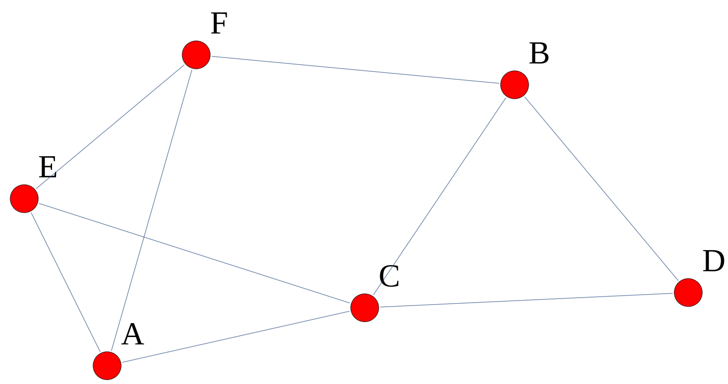
Below are two random networks. See if you can write down the adjacency matrix for both of them

1. This network has a loop at the A node. That means the node is connected **to itself**. How would you represent that in the adjacency matrix?



$$\underline{\mathbf{A}} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left(\begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \end{array} \right) \end{array}$$

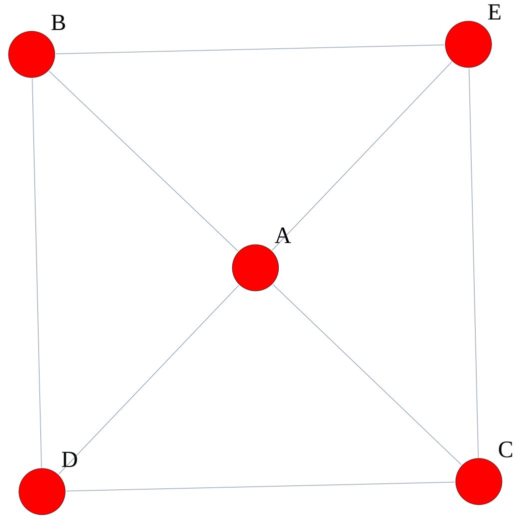
2. Lets try something a bit bigger



$$\underline{\mathbf{A}} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \left(\begin{array}{cccccc} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array} \right) \end{array}$$

4 Who Cares?

Good question. Once you have the adjacency matrix, there are a number of things to you do. For instance, lets say I want to know if I can get from A to E in this network:



1. How many ways can I go from $A \to E$ using just one hop?

2. What about with two hops?

3. What if I take 3 hops?

Now you can count these routes all up by hand, but that gets boring (what if I asked how many ways you could get there with 20 hops?). Instead, you can use the adjacency matrix!

1. Write down the Adjacency Matrix for the graph:

$$\underline{\mathbf{A}} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} A \quad B \quad C \quad D \quad E \\ \left(\begin{array}{ccccc} \quad & \quad & \quad & \quad & \quad \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right) \end{array}$$

2. Now, write down the **square** of the adjacency matrix, $\underline{\mathbf{A}}^2 = \underline{\mathbf{A}} \times \underline{\mathbf{A}}$

$$\underline{\mathbf{A}}^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left(\begin{array}{ccccc} & & & & \\ \\ \\ \\ \\ \end{array}\right) \end{array}$$

Look at the $A \to E$ number. Is it the same number you wrote down above for two hops?

3. I'll save you some trouble, the **cube** of the adjacency matrix, $\underline{\mathbf{A}}^3 = \underline{\mathbf{A}} \times \underline{\mathbf{A}} \times \underline{\mathbf{A}}$ is

$$\underline{\mathbf{A}}^3 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left(\begin{array}{ccccc} 8 & 8 & 8 & 8 & 8 \\ 8 & 4 & 4 & 8 & 8 \\ 8 & 4 & 4 & 8 & 8 \\ 8 & 8 & 8 & 4 & 4 \\ 8 & 8 & 8 & 4 & 4 \end{array}\right) \end{array}$$

Now look at the $A \to E$ number again. Is that the same number you got for three hops?

This is what we're after: **you can figure out how many ways you can get from one point to another in x jumps by raising the adjacency matrix to the x power**.